"The notion of number, integral or fractional, has been placed upon a basis entirely independent of measurable magnitude, and pure analysis is regarded as a scheme which deals with number only, and has, *per se*, no concern with measurable quantity. Analysis thus placed upon an arithmetical basis is *characterized by the rejection of all appeals to our special intuitions of space, time and motion, in support of the possibility of its operations.* . . ."

"By this conception of the domain of number the root difficulty of the older analysis as to the existence of a limit is turned, each number of the continuum being really defined in such a way that it itself exhibits the limit of certain classes of convergent sequences. . . . It should be observed that the criterion for the convergence of an aggregate is of such a character that no use is made in it of infinitesimals, definite finite numbers alone being used in the test. . . .

"This [old] intuitive notion of the continuum appears to have as its content the notion of unlimited divisibility, the facts that, for instance, in the linear continuum we can within any interval PQ find a smaller one, $P'Q'$, that this process may be continued as far as the limits of our perception allow, and that we are unable to conceive that even beyond the limits of our perception the process of divisibility in thought can come to an end. However, the modern discussions as to the nature of the arithmetic continuum have made it clear that this property of unlimited divisibility, or connexity, is only one of the distinguishing characteristics of the continuum, and is insufficient to mark it off from other domains which have the like property. The aggregate of rational numbers, or of points on a straight line corresponding to such numbers, possess this property of connexity in common with the continuum, and yet it is not continuous." . . .

"The other property of the aggregate which is characteristic of the continuum, is that of being, in the technical language of the theory of aggregates (Mengenlehre) perfect: the meaning of this is that all the limits of the converging sequences of numbers or parts belonging to the aggregate themselves belong to the aggregate; and, conversely, that every number or point of the aggregate can be exhibited as the limit of such a sequence. . . .

". . . the latter property of the continuum, which was not brought to light by those who took the intuitive continuum as a sufficient basis, is in some respects the more absolutely essential property for the domain of a function which is to be submitted to the operation of the calculus."

"In order to exhibit the way in which transfinite ordinal numbers are required when we deal with non-finite aggregates, I propose to refer to a well-known paradox of Achilles and the tortoise. . . . Let us indicate the successive positions of Achilles referred to, by the ordinal numbers 1, 2, 3, . . . suffixed to the letter A, so that $A_1\ A_2\ A_3\ \ldots$ represent the positions of Achilles. . . . These points $A_1\ A_2$

$$\begin{array}{cccccccccc} & B_2 & & B_3 & B_4 & & & & & \\ \hline A_1 & & A_2 & A_3 & A_4 & A_\omega & A_{\omega+1} & A_{\omega+2} & A_{\omega 2} & \end{array}$$

A_3 . . . have a limiting point, which represents the place where Achilles actually catches the tortoise. The limiting point is not contained in the sets of points $A_1\ A_2\ A_3\ \ldots$; if we wish to represent it, we must introduce a new symbol ω, and denote the point by this number. It does not occur in the series 1, 2, 3, . . . but is preceded by all of these numbers, and yet there is no number immediately preceding it; it is the first of a new series of numbers."

Hobson proceeds to show how a finite number can maintain itself against a transfinite ordinal number, by showing that $\omega = 1 + \omega$, but $\omega + 1 > \omega$; the commutative law in addition is seen to fail. He brings out the necessity for the introduction of transfinite numbers for the representation of the limit which is not itself contained within the region of the convergent process. Hobson's exposition represents an explanation which recent developments of mathematics offer of the "Dichotomy" and the "Achilles." No doubt, some readers might have desired a fuller exposition of details. It will be noticed that the time-element was not considered by Hobson at all. The Dedekind and Georg Cantor theories of the continuum do not involve the element of time. But how is it possible to ignore time in questions involving motion? In the first place it is pointed out by Cantor[1] that the continuum is a much more primitive and general

[1] *Grundlagen einer allgemeinen Mannichfaltigkeitslehre*, von Georg Cantor, Leipzig, 1883, p. 29.

concept than the concept of time, that the theory of the continuum is needed for a clear exposition of time or of any independent variable, that time cannot be considered as the measure of motion; on the contrary, time is measured by motion—the motions of heavenly bodies, the motions of the hands of a watch or clock, the displacement of sand in the hour glass. In the second place the consideration of time is not needed at the critical point where the ability of Achilles to overtake the tortoise is under consideration. Suppose the tortoise has an initial start of 10 ft. and that it travels 1 ft. per second, while Achilles travels 10 ft. per second. Forming the series A, whose terms represent each the distance Achilles travels to come up to the place where the tortoise was at the beginning of the time interval under consideration, and letting the series T represent these time-intervals, we have

$$A \qquad 10 + 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots,$$

$$T \qquad 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \cdots.$$

Both geometric infinite series are convergent; the sum of the terms of each series approaches a finite number as a limit. Now comes the ever-present, delicate question whether the sum actually *reaches* its limit. It is to be observed that this question arises in each series, that one series does not help the other. If the sum of A reaches its limit, so does the sum of T, but the possibility of the sum of A reaching its limit is a consideration independent of T. In this sense the consideration of time does not enter the critical part in the explanation of the "Achilles." Whether the sum of A reaches its limit or not is a matter of pure assumption on our part. If the limiting value $11\frac{1}{9}$ ft. is assumed to be included in the aggregate of numbers which the distance-variable may take, then of course the variable reaches its limit; if $11\frac{1}{9}$ ft. is not assumed as a value which the variable may take, then of course the limit is not reached. It is here that we must receive a suggestion from our sensuous observations; we know from our knowledge of motion as supplied to us by our senses that Achilles travelling with a uniform finite velocity in the same direction will within a finite time reach the distance $11\frac{1}{9}$ ft. from his starting point. This information, supplied to us by our senses, enables us to choose, of the two possible alternative assumptions offered by theory (as mentioned above), the one which makes the variable sum A conform with the known sensuous phenomena. On this assumption the region of the convergent process has a limit which is contained in the aggregate of values the variable can take, and, as explained by Hobson, the limit is *reached by the variable.* Viewed from the standpoint of the theory of infinite aggregates and of the Georg Cantor conception of the continuum, the "Achilles" is almost a self-evident proposition. Sensuous knowledge suggests that we make the aggregate of values of the variable distance travelled by Achilles a *perfect* aggregate; then theory tells us that in a perfect aggregate every converging process has a limit which is reached.

Interesting remarks on the Georg Cantor continuum are found also in Hobson's *Theory of Functions of a Real Variable*, Cambridge, 1907, p. 51:

"The term 'arithmetic continuum' is used to denote the aggregate of real numbers, because it is held that the system of numbers of this aggregate is adequate for the complete analytical representation of what is known as continuous magnitude. The theory of the arithmetic continuum has been criticised on the ground that it is an attempt to find the continuous within the domain of number, whereas number is essentially discrete. Such an objection presupposes the existence of some independent conception of the continuum, with which that of the aggregate of real numbers can be compared. At the time when the theory of the arithmetic continuum was developed the only conception of the continuum which was extant was that of the continuum as given by intuition: but this, as we shall show, is too vague a conception to be fitted for an object of exact mathematical thought, until its character as a pure intuitional datum has been modified by exact definitions and axioms."

It will be seen as we proceed that the objection to the Georg Cantor continuum, to which Hobson refers, is frequently made. It is the general objection that a line which is continuous cannot possibly be constructed out of mathematical points, external to each other. Perhaps this general objection which naturally suggests itself at the very start has discouraged non-mathematicians from going to the trouble of studying the Cantor continuum with the care necessary for its comprehension. Philosophers who have subjected themselves to such study have been amply repaid for their labor. They have found it to be a device of the understanding "whereby we give conceptual unity and an invisible connectedness to certain types of phenomenal facts which come to us in a discrete form and in a confused variety."[1]

The other important shift in the point of view, made by the creators of the modern linear continuum, was the rejection of all infinitesimals, that is of quantities which do not obey the Archimedian postulate. This postulate says that if a and b are two numbers (not zero), such that $a < b$, then it is always possible to find a finite integer n so that $na > b$. The infinitesimal which had been the subject of many controversies and was regarded by many as containing an element of mysticism, was banished by Weierstrass and Cantor from their mathematical concepts. In former years the infinitesimal was considered as necessary in the explanation of the linear continuum. Johann Heinrich Lambert wrote to Holland in a letter of April 7, 1766, on the "angle of contact" as follows:[2]

"Do you believe, my dear Sir, that one can dispense with the concept of the infinitely small in the concept of continuity? . . . Continuity demands that this variation be less than every assignable quantity. It is thus impossible to estimate this variation by a finite quantity, and equal to 0 it can not be either. There seems therefore nothing left than to say that the change in direction is infinitely small."

The impossibilities of one generation often become the possibilities of a succeeding generation. Weierstrass's banishment of the infinitely small has found wide following; the old-time infinitesimal is no longer needed in explaining the continuum. The rejection of the infinitely small is looked upon by such mathe-

[1] H. Poincaré, *The Foundations of Science*, transl. by G. B. Halsted, New York, 1913, introduction by Josiah Royce, p. 16.

[2] *J. H. Lamberts deutscher gelehrter Briefwechsel*, Vol. I, Berlin, 1781, p. 141.

matical philosophers and logicians as Bertrand Russell[1] and A. N. Whitehead[2] as steps toward greater mathematical rigor. It must be emphasized, however, that the school of Weierstrass has not found universal recognition; there are modern champions of the infinitely small, chief among whom is the Italian mathematician Giuseppe Veronese. They insist that the Cantor continuum is not the only possible non-contradictory continuum and proceed to construct a higher and more involved, non-archimedean, continuum in which infinitely small distances are given. This is not the place for attempting a minute statement of the controversy between the two schools; the controversy, by the way, has no national aspect. There have been followers of Veronese in Germany (for instance, Stolz, Max Simon), and followers of Weierstrass and G. Cantor in Italy (for instance, Peano). So far as we have noticed, the Zeno arguments have not been studied and given explicit treatment on the basis of the Veronese continuum.[3] In America C. S. Peirce has adhered to the idea of infinitesimals in the declaration: "The illumination of the subject by a strict notation for the logic of relatives had shown me clearly and evidently that the idea of an infinitesimal involves no contradiction."[4] Apparently, before he had acquired familiarity with the writings of Dedekind and Georg Cantor, C. S. Peirce had firmly recognized that for infinite collections the axiom, that the whole is greater than its part, does not hold.

[To be continued]

ON NAPIER'S FUNDAMENTAL THEOREM RELATING TO RIGHT SPHERICAL TRIANGLES.

By ROBERT MORITZ, University of Washington.

In view of the recent celebration of the tercentenary of the publication of Napier's greatest work, the "Mirifici logarithmorum canonis descriptio," it is highly fitting that his rule for the circular parts should be rescued from the rubbish heap of mnemotechnics and be assigned its proper place as the most

[1] See, for instance, his article in the *International Monthly*, Vol. 4, 1901, p. 84 and seq.

[2] A. N. Whitehead, *Introduction to Mathematics*, New York and London, 1911, pp. 156, 226–229.

[3] References to this controversy are as follows: G. Veronese, *Grundzüge der Geometrie von mehreren Dimensionen*, übersetzt v. A. Schepp, Leipzig, 1894, Anhang, p. 631–701; Max Simon, "Historische Bemerkungen über das Continuum, den Punkt und die Gerade Linie," *Atti del IV. Congresso Internazionale dei matematici*, Roma, 1908, pp. 385–390; G. Cantor's letter to Vivanti, *Rivista di mat.* V, 104–108; G. Cantor's letter to Peano, *Rivista di mat.* V, 108–109; G. Cantor, "Zur Begründung der Transfiniten Mengenlehre I," *Mathematische Annalen*, Vol. 46, 1895, page 500; Frederico Enriques, *Probleme der Wissenschaft*, 2. Teil, übersetzt von K. Grelling, Leipzig und Berlin, 1910, pp. 324–329. An able discussion of infinity, infinitesimals and the continuum is given by Jcsiah Royce, a philosopher familiar with mathematical thought, in his *The World and the Individual*, New York, 1900, pp. 505–560. See also G. Cantor, "Mitteilungen zur Lehre vom Transfiniten" in *Zeitsch. für Philosophie u. Philosophische Kritik*, Vol. 91, Halle, 1887, p. 113; O. Stolz in *Mathematische Annalen*, Bd. XVIII, p. 699, also in *Berichte des naturw.-medizin. Vereins in Innsbruck*, Jahrgänge 1881–82 und 1884, also in *Vorlesungen über allgem. Arithm.*, Leipzig, 1. Theil, 1885, p. 205.

[4] C. S. Peirce, "The Law of Mind" in *The Monist*, Vol. 2, 1892, p. 537.

beautiful theorem in the whole field of elementary trigonometry. It is one of the strange vicissitudes of fortune that the elegant proof which was clearly indicated by Napier himself in the fourth chapter of the second book of the "descriptio" and rediscovered by Lambert[1] and Ellis[2] should nevertheless have remained generally unknown to writers on trigonometry in the nineteenth century and that even to this day the impression generally prevails that Napier's rules are nothing more than mnemonic devices whose utility as an instrument may well be questioned. Excepting two recent texts[3] one seeks in vain for any intimation that Napier's rules for the circular parts have any other than an inductive basis.

Thus as omniscient a writer as DeMorgan in his *Spherical Trigonometry* speaks of Napier's rules as "mnemonical formulas" and expresses his conviction that they "only create confusion instead of assisting the memory." Chauvenet (*Plane and Spherical Trigonometry*, 1891) after developing the ten formulas for the solution of right spherical triangles, says: "By putting these ten rules under a different form, Napier contrived to express them all in two rules, which, though artificial, are very generally employed as aids to the memory." In like tenor Newcomb (*Trigonometry*, 1893): "The six preceding formulæ, which may be found difficult to remember, have been included by Napier in two precepts of remarkable simplicity, and easily remembered" and the same view is reiterated in the recent work of Bôcher and Gaylord (*Trigonometry*, 1914) in the words "Formulas 1–10 may be collected into a very compact and convenient form by means of a rule formulated by John Napier. The student should prove that these rules are correct by applying them in succession to all five parts of the figure."

Nor is the impression that Napier's rules have no other than an inductive basis limited to writers of textbooks on trigonometry. Cajori in his *History of Mathematics* states that "Napier's Rule of circular parts is perhaps the happiest example of artificial memory that is known," thus putting this remarkable achievement in deduction on a par and in direct competition with the famous mnemonic hexameter of the logicians,

> "Barbara, Celarent, Darii, Ferioque prioris.
> Cesare, Camestres, Festino, Baroko secundae.
> Tertia Darapti, Disamis, Datisi, Felapton,
> Bokardo, Ferison habet. Quarta insuper addit
> Bramantip, Camenes, Dimaris, Fesapo, Fresison,"

of which Hamilton said "there are few human inventions which display a higher ingenuity."

Even as high and recent an authority as E. W. Hobson in his article on trigonometry in the eleventh edition of the Encyclopedia Britannica dismisses the whole matter with the words "Napier gave mnemonical rules for remembering" the right spherical triangle relations.

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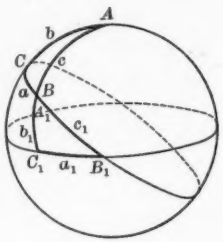
[1] *Beiträge zur Mathematik*, I (1765), p. 375 et seq.

[2] *The Mathematical and other Writings of Robert Leslie Ellis* (1863), p. 328 et seq.

[3] Leathem-Todhunter: *Spherical Trigonometry*, Macmillan (1907). Moritz: *Spherical Trigonometry*, Wiley and Sons (1914).

In view of this generally prevailing misconception, it may be timely to present in this place an elementary proof of Napier's theorem which will in a sense supplement Professor E. O. Lovett's prior note[1] on the same subject.

Let $\Delta = ABC$ be a right spherical triangle whose parts taken in counter-clockwise order beginning with a side adjacent to the right angle are a, B, c, A, b. With A as a pole construct the arc of a great circle intersecting the arcs CB produced and AB produced in B_1 and C_1 respectively. A second right triangle $\Delta_1 = A_1B_1C_1$ is thus formed whose parts a_1, B_1, c_1, A_1, b_1 are



$$a_1 = \overline{A},\ B_1 = \overline{b},\ c_1 = \bar{a},\ A_1 = B,\ b_1 = \overline{c},$$

where

$$\overline{A} = 90° - A, \qquad \overline{b} = 90° - b, \qquad \bar{a} = 90° - a, \text{ etc.}$$

Thus every triangle Δ whose circular parts are $a\ \overline{B}\ \overline{c}\ \overline{A}\ b$, leads to a second triangle Δ_1 whose circular parts are $\overline{A}\ b\ a\ \overline{B}\ \overline{c}$, and this likewise to a third Δ_2 whose circular parts are $\overline{B}\ \overline{c}\ \overline{A}\ b\ a$, and this again to a fourth Δ_3 whose circular parts are $b\ a\ \overline{B}\ \overline{c}\ \overline{A}$, and this in turn to a fifth Δ_4 whose circular parts are $\overline{c}\ \overline{A}\ b\ a\ \overline{B}$. The fifth triangle, Δ_4, leads to the first, Δ, thus completing the cycle.

The circular parts of these five triangles are thus shown to be the same for all. Furthermore, if we call the hypotenuse of a right triangle its middle part, this name may be applied to each of the circular parts, for each of the circular parts $a, \overline{B}, \overline{c}, \overline{A}, b$, is the hypotenuse of some one of the five triangles $\Delta, \Delta_1, \Delta_2, \Delta_3, \Delta_4$. It follows that the relation between any middle part and its two adjacent (contiguous) parts applies to every middle part and its two adjacent parts, and again, that the relation between any middle part and its two opposite (non-contiguous) parts applies to every middle part and its two opposite parts.

Now from the first triangle ABC we find from the law of cosines (or otherwise) that $\cos c = \cos a \cos b$, that is $\sin \overline{c} = \cos a \cos b$, and on eliminating from this equation a and b by the law of sines (or otherwise) we obtain $\cos c = \cot A \cot B$, that is $\sin \overline{c} = \tan \overline{A} \tan \overline{B}$. This proves *Napier's Theorem: The sine of any circular part is equal to the product of the cosines of the opposite parts and to the product of tangents of the adjacent parts.*

[1] *Bulletin of the American Mathematical Society*, 1898, p. 552.

AN INTERPOLATION FORMULA FOR POISSON'S EXPONENTIAL BINOMIAL LIMIT.

By E. C. MOLINA, New York City.

Number LII of the very valuable "Tables for Statisticians and Biometricians" recently edited by Karl Pearson gives the values of Poisson's exponential binomial limit,

$$P(c, a) = \sum_{s=c}^{\infty} \frac{\epsilon^{-a} a^s}{\lfloor s}$$

for integral values of a from 1 to 30 inclusive.

Suppose we want the value of

$$P(c, a + \delta), \qquad (\delta = 0.1, 0.2, \cdots 0.9).$$

By successive partial integration it is easy to see that

$$P(c, a) = \frac{1}{\lfloor c - 1} \int_0^a x^{c-1} \epsilon^{-x} dx. \tag{1}$$

Therefore,

$$P(c, a + \delta) = P(c, \delta) + \frac{1}{\lfloor c - 1} \int_\delta^{a+\delta} x^{c-1} \epsilon^{-x} dx. \tag{2}$$

Now

$$\frac{1}{\lfloor c - 1} \int_\delta^{a+\delta} x^{c-1} \epsilon^{-x} dx = \frac{\epsilon^{-\delta} \delta^{c-1}}{\lfloor c - 1} \int_0^a \left(1 + \frac{y}{\delta}\right)^{c-1} \epsilon^{-y} dy$$

$$= \sum_{u=1}^{c} \frac{\epsilon^{-\delta} \delta^{c-u}}{\lfloor c - u \rfloor u - 1} \int_0^a y^{u-1} \epsilon^{-y} dy \tag{3}$$

$$= \sum_{u=1}^{c} \left[\frac{\epsilon^{-\delta} \delta^{c-u}}{\lfloor c - u} P(u, a) \right] = \sum_{t=0}^{c-1} \left[\frac{\epsilon^{-\delta} \delta^t}{\lfloor t} P(c - t, a) \right].$$

Thus, finally

$$P(c, a + \delta) = P(c, \delta) + \sum_{t=0}^{c-1} \left[\frac{\epsilon^{-\delta} \delta^t}{\lfloor t} P(c - t, a) \right]. \tag{4}$$

As stated above, the values of $P(c - t, a)$ for $a > 30$ are given in Pearson's Table LII. Also, in the first section of Pearson's Table LI are given the values of $\epsilon^{-\delta} \delta^t / \lfloor t$.

For the practical application of formula (4) we need, however, the values of $P(c, \delta)$. These are given in the short tables which follow.

TABLE GIVING VALUES OF $P(c, \delta)$.

c	δ									
	0.1	0 2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	.....	.....	.....	.....	.....	.....	.....	.....	.....	.....
1	.095163	.181269	.259182	.329680	.393469	.451188	.503415	.550671	.593431	.63212
2	.004679	.017523	.036936	.061552	.090204	.121901	.155805	.191208	.227518	.26424
3	.000155	.001148	.003600	.007926	.014388	.023115	.034142	.047423	.062857	.08030
4	.000004	.000057	.000266	.000776	.001752	.003358	.005753	.009080	.013459	.01899
5	.000000	.000002	.000016	.000061	.000172	.000395	.000786	.001411	.002344	.00366
6	.....	.000000	.000001	.000004	.000014	.000039	.000090	.000184	.000343	.00059
7	.....	.....	.000000	.000000	.000001	.000003	.000009	.000021	.000043	.00008
8	.....	.....	.....	.....	.000000	.000000	.000001	.000002	.000005	.00001
9	.....	.....	.....	.....	.....	.....	.000000	.000000	.000000	.00000

Example.—Find value of P (20, 14.8).
Here $c = 20$, $a = 14$, $\delta = .8$.

t	$e^{-\delta\delta t}/t$, Table LI, page 113.	$P(c-t, a)$, Table LII, page 123.	$\frac{e^{-\delta\delta t}}{t} P(c-t, a)$.
0	.449329	.07650	.034374
1	.359463	.11736	.042186
2	.143785	.17280	.024846
3	.038343	.24408	.009359
4	.007669	.33064	.002536
5	.001227	.42956	.000527
6	.000164	.53555	.000088
7	.000019	.64154	.000012
8	.000002	.73996	.000001
9	.000000		.113929

$P(c, \delta) = P(20, 0.8) = .000000$
$P(c, a + \delta) = P(20, 14.8) = .11393$

January 19, 1915.

A SIMPLE SOLUTION OF THE DIOPHANTINE EQUATION $U^3 = V^3 + X^3 + Y^3$.

By J. W. NICHOLSON, Louisiana State University.

There are many sets of four integers such that the cube of the first is equal to the sum of the cubes of the other three. For examples,

$$6^3 = 5^3 + 4^3 + 3^3, \tag{1}$$

$$9^3 = 8^3 + 6^3 + 1^3, \tag{2}$$

$$19^3 = 18^3 + 10^3 + 3^3. \tag{3}$$

The purpose of this paper is to deduce a formula by which an unlimited number of such sets may be obtained. It is an interesting question and has frequently engaged the attention of mathematicians. The problem has been treated by Diophantus, Fermat, Euler, and others. Notwithstanding their work, it is believed that the following solution may be of interest. Furthermore, by sub-

stantially the same method of deduction we may solve many other problems in indeterminate analysis, notably the following,

$$r^3 = s^3 + t^3 + u^3 + v^3,$$

$$r^3 = s^3 + t^3 + u^3 + v^3 + x^3,$$

and so on, for any greater number of terms whatever.

Problem. Given one set of four integers m, n, p, r such that

$$m^3 = n^3 + p^3 + r^3, \tag{4}$$

to find a formula for an indefinite number of such sets. Evidently,

$$(my)^3 = (ny)^3 + (py)^3 + (ry)^3 \tag{5}$$

where y may have any value whatever.

Assume

$$(my - bx)^3 = (ny - bx)^3 + (py - ax)^3 + (ry + ax)^3, \tag{6}$$

in which a and b are arbitrary constants.

We wish now to find such values of x and y as will make (6) an identity. Expanding in (6) and reducing, we find

$$(mb^2 - nb^2 - pa^2 - ra^2)x = (m^2b - n^2b - p^2a + r^2a)y.$$

We may therefore take

$$y = mb^2 - nb^2 - pa^2 - ra^2; \quad x = m^2b - n^2b - p^2a + r^2a.$$

Substitute these values for x and y in (6) and we obtain the required formula:

$$\begin{aligned}[(mp + mr)a^2 - (p^2 - r^2)ab + (mn - n^2)b^2]^3 \\ = [(np + nr)a^2 - (p^2 - r^2)ab + (m^2 - mn)b^2]^3 \\ + [(pr + r^2)a^2 + (m^2 - n^2)ab - (mp - np)b^2]^3 \\ + [(p^2 + pr)a^2 - (m^2 - n^2)ab - (mr - nr)b^2]^3.\end{aligned} \tag{A}$$

For example we may make $m = 6$, $n = 5$, $p = 4$, $r = 3$, and obtain

$$\begin{aligned}(42a^2 - 7ab + 5b^2)^3 = (35a^2 - 7ab + 6b^2)^3 \\ + (28a^2 - 11ab - 3b^2)^3 + (21a^2 + 11ab - 4b^2)^3,\end{aligned} \tag{B}$$

which holds for all values of a and b.

Thus, for $a = 10$ and $b = 1$, $4135^3 = 3436^3 + 2687^3 + 2206^3$.

Again, for $a = 1$ and $b = 4$, $103^3 = 94^3 + 64^3 - 1^3$.

BOOK REVIEWS.

EDITED BY W. H. BUSSEY, University of Minnesota.

Vocational Mathematics. By WILLIAM H. DOOLEY. D. C. Heath and Co., Boston, 1915. 358 pages. $1.00.

The author of this book is the principal of the Technical High School of Fall River, Mass. He states in the preface that, in his ten years of experience in organizing and conducting intermediate and secondary technical schools, he has noticed the inability of the regular teachers in mathematics to give the pupils the training in commercial and rule of thumb methods of solving mathematical problems that are so necessary in everyday life. He says that the pupils graduate from the course in mathematics without being able to "commercialize" or apply their mathematical knowledge in such a way as to meet the needs of trade and industry. It was to overcome this difficulty that he wrote this book. It is in the following ten parts, whose names sufficiently indicate the scope of the work: Review of arithmetic; carpentering and building; sheet metal work; bolts, screws and rivets; shafts, pulleys and gears; plumbing and hydraulics; steam engineering; electrical work; mathematics for machinists; textile calculations. There is an appendix of about 40 pages devoted to the metric system, graphs, formulas, logarithms, trigonometry, and tables of various kinds.

W. H. BUSSEY.

A Review of Algebra. By ROMEYN HENRY RIVENBURG. American Book Co., New York, 1914. 80 pages.

This is a book of problems prepared by the head of the department of mathematics of the Peddie Institute, Hightstown, N. J. It begins with a seven-page outline of elementary and intermediate algebra which contains important definitions, special rules for multiplication and division, cases in factoring, etc. It is designed for senior high-school students who need a thorough reivew of algebra in order to prepare for college entrance examinations and for effective work in the freshman year in college. The whole scheme of the book is ordinarily to have a page of problems represent a day's work. It includes quadratics, simultaneous quadratics, the progressions and the binomial theorem. There are twenty-three pages of actual college entrance examinations at the end of the book. The author states in the preface that the student is expected to use his regular text book in algebra for reference, as he would use a dictionary,—to recall a definition, a rule, or a process that he has forgotten.

W. H. BUSSEY.

An Introduction to Laboratory Physics. By LUCIUS TUTTLE. Jefferson Laboratory of Physics, Philadelphia, 1915. 150 pages.

This book is essentially a revision of the mimeographed direction sheets that have been used in the first part of the laboratory course given by the author at

Jefferson Medical College. It does not cover the ground of the usual laboratory manual of physics, but is intended to precede the use of any one of them in a course of physical measurements. It contains chapters on weights and measures, angles and circular functions, accuracy and significant figures, logarithms, small magnitudes, the slide rule, graphic representation, graphic analysis, the principle of coincidence, measurements and errors, statistical methods, deviation and dispersion, the weighting of observations, criteria of rejection, and least squares and various errors.

W. H. BUSSEY.

UNIVERSITY OF MINNESOTA.

Resistance of Materials. For beginners in Engineering. By S. E. SLOCUM, Professor of Applied Mathematics in the University of Cincinnati. Ginn & Company, Boston, 1914. $2.00.

The chief feature which distinguishes this volume from other American text-books on the same subject, as stated by the author in his preface, is that the principle of moments is used consistently throughout in place of the usual calculus processes.

The subject matter is presented in a clear and concise manner and is written so the book may be used by students at the same time they are taking courses in calculus or even before taking such courses. This feature makes the book available for trade or architectural schools where no calculus is taught.

The text has been divided into the following fourteen sections: stress and deformation; first and second moments; bending moment and shear diagrams; strength of beams; deflection of cantilever and simple beams; continuous beams; restrained or built-in beams; columns and struts; torsion; spheres and cylinders under uniform pressure; flat plates; riveted joints and connections; reinforced concrete; simple structures.

The author has used the principle of integration freely throughout the text dealing with the small parts, "elements," represented by Δ and using the symbol Σ instead of $\int$. Such a ratio as $\Delta M/\Delta x$ has been defined as the "rate of change," instead of the usual dM/dx of the calculus; also in finding the area of the moment diagram the expression $\Sigma M \cdot \Delta x$ replaces the $\int M dx$ of the calculus.

Instead of deducing the equation of the elastic curve of a loaded beam and from it the slope and deflection at any section, the author has used for the deflection, $d = 1/EI$ (static moment of the moment diagram), and for the slope, $\tan \varphi = A/EI$, where A represents the area of the moment diagram between any two points in question. These equations give simple solutions in most cases. In dealing with the continuous beam and the restrained or built-in beam, the necessary formulas have been deduced by considering the effect of the loads acting separately. The sections dealing with torsion, spheres and cylinders under uniform pressure, and flat plates, contain the usual formulas for these cases.

The feature of the book that appealed most strongly to the reviewer was the

large number of well graded applications that have been included. The solving of these applications cannot fail to give the student a better understanding of the fundamentals involved and at the same time should tend to stimulate his interest in the subject.

WILLIAM A. JOHNSTON.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND R. P. BAKER.

PROBLEMS FOR SOLUTION.

ALGEBRA.

438. Proposed by WALTER C. EELLS, U. S. Naval Academy, Annapolis, Maryland.

In Hardy's *Pure Mathematics* (page 14, Nos. 2, 3) occurs the problem: "Show that if m/n is a good approximation to $\sqrt{2}$, then $(m + 2n)/(m + n)$ is a better one, and that the errors in the two cases are in opposite directions, *e. g.*, 1/1, 3/2, 7/5, 17/12, 41/29, 99/70, ···." Find (*a*) other approximations for $\sqrt{2}$ of same type, *i. e.*,

$$\frac{m'}{n'} = \frac{am + bn}{cm + dn}, \qquad (a, b, c, d, m, n, \text{integers}).$$

(*b*) Similar approximations for the square roots of other integers.

439. Proposed by A. M. KENYON, Purdue University.

If k, n are natural numbers, $n > 2k$, show that

$$\frac{2^k}{\lfloor k} \sum_{i=0}^{I\left(\frac{n+1}{2}\right)} \frac{1}{\lfloor 2i+1 \rfloor \lfloor n-k-2i}} = \frac{2^n}{\lfloor n+1} \sum_{i=0}^{k} \binom{n-i}{n-k},$$

where $I(n/2)$ denotes the integral part of $n/2$ and $\binom{n}{k}$ is the coefficient of x^k in $(1 + x)^n$.

440. Proposed by W. D. CAIRNS, Oberlin College.

n being a positive integer, find the sum of the series

$$n^2 + 4(n-1)^2 + 2(n-2)^2 + 4(n-3)^2 + 2(n-4)^2 + \cdots, \tag{1}$$

where the succeeding coefficients are alternately 4 and 2; or, more generally, the series

$$an^2 + b(n-1)^2 + a(n-2)^2 + b(n-3)^2 + b(n-3)^2 + \cdots. \tag{2}$$

L'Intermédiaire, July, 1913.

GEOMETRY.

469. Proposed by W. F. FLEMING, Chicago, Ill.

A pole whose length is l stands vertically against a vertical wall. A spider is at each end of the pole. The pole is drawn out from the wall in such a way that its upper end moves down the wall at a uniform rate. At the same time that the pole begins to move, the spiders begin to travel toward each other at rates equal to the rates at which the respective ends move. Determine the equations of the paths of the two spiders, in space.

470. Proposed by ROBERT E. MORITZ, University of Washington.

Prove that

$$\theta = \left(\lambda + \frac{q}{p}\mu\right)\pi, \qquad (\lambda = 1, 2, 3, \cdots, q-1;\ \mu = 0, 1, 2, \cdots, p-1)$$

and

$$\theta = (2\lambda - 1)\frac{\pi}{2} + \frac{q}{p}(4\mu \pm 1)\frac{\pi}{2}, \qquad (\lambda = 1, 2, 3, \cdots, \frac{q-1}{2};\ \mu = 0, 1, 2, \cdots, p-1);$$

determine the same set of points on the curve

$$\rho = a \cos \frac{p}{q}\theta,$$

where p and q are two odd integers without a common factor, and a is any constant.

471. Proposed by C. N. SCHMALL, New York City.

In the ellipse $x^2/a^2 + y^2/b^2 = 1$, an *equilateral* hexagon is inscribed with two sides parallel to the major axis. In the major auxiliary circle the same thing is done. If H_1 and H_2 be the sides of the hexagons, and e the eccentricity of the ellipse, show that $H_1 : H_2 :: 4 - 2e^2 : 4 - e^2$

CALCULUS.

390. Proposed by WILSON L. MISER, University of Minnesota.

Show that the triangle whose area is a constant and whose perimeter is a minimum is equilateral.

391. Proposed by H. B. PHILLIPS, Massachusetts Institute of Technology.

If $0 < \lambda < 1$ and $0 < x < \pi$, show that the function $(\sin \lambda x)/(\sin x)$ increases as x increases.

392. Proposed by HORACE OLSON, Student at The University of Chicago.

Two right circular cylinders of radii a and b respectively, are placed so that their axes intersect at right angles. Find the volume common to them.

MECHANICS.

314. Proposed by C. N. SCHMALL, New York City.

A rectangular box of height h, and having a plane mirror for its bottom, contains a quantity of water of unknown height x. In the lid are two small apertures distant $2a$ from each other. A ray of light entering one aperture with an angle of incidence ϕ, emerges, after refraction and reflection, through the other aperture. If μ be the index of refraction of water, show that the height of the water is

$$x = \frac{(h \tan \phi - a)}{\left[\tan \phi - \dfrac{\sin \phi}{(\mu^2 - \sin^2 \phi)^{\frac{1}{2}}}\right]}.$$

SOLUTIONS OF PROBLEMS.

ALGEBRA.

427. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

If $r \sin(\theta + \alpha) = m$, and $r \cos(\theta + \beta) = n$, show that

$$r = \frac{\sqrt{m^2 + n^2 - 2mn \sin(\alpha - \beta)}}{\cos(\alpha - \beta)}.$$

SOLUTION BY J. H. KELLOGG, Oberlin College.

From trigonometry, using only the positive square root, we know that

$$\sin(A+B) = \sqrt{(\sin A\cos B + \cos A \sin B)^2}.$$

$$= \sqrt{\sin^2 A(1-\sin^2 B) + (1-\sin^2 A)\sin^2 B + 2\sin A\cos A\sin B\cos B}$$

$$= \sqrt{\sin^2 A + \sin^2 B + 2\sin A\sin B(\cos A\cos B - \sin A\sin B)}$$

$$= \sqrt{\sin^2 A + \sin^2 B + 2\sin A\sin B\cos(A+B)}.$$

From the given equations we have

$$\sin(\theta+\alpha) = \frac{m}{r} \qquad \text{and} \qquad \sin\left[\frac{\pi}{2} - (\theta+\beta)\right] = \frac{n}{r}.$$

Hence letting

$$A = (\theta+\alpha) \qquad \text{and} \qquad B = \left[\frac{\pi}{2} - (\theta+\beta)\right],$$

we have

$$(A+B) = \left[\frac{\pi}{2} + (\alpha-\beta)\right].$$

Substituting, we have

$$\sin\left[\frac{\pi}{2} + (\alpha-\beta)\right]$$

$$= \sqrt{\sin^2(\theta+\alpha) + \sin^2\left[\frac{\pi}{2} - (\theta+\beta)\right] + 2\sin(\theta+\alpha)\sin\left[\frac{\pi}{2} - (\theta+\beta)\right]\cos\left[\frac{\pi}{2} + (\alpha-\beta)\right]},$$

or

$$\cos(\alpha-\beta) = \sqrt{\frac{m^2}{r^2} + \frac{n^2}{r^2} - 2\frac{m}{r}\frac{n}{r}\sin(\alpha-\beta)}.$$

Solving for r, we have

$$r = \frac{\sqrt{m^2+n^2-2mn\sin(\alpha-\beta)}}{\cos(\alpha-\beta)}.$$

Also solved by B. J. Brown, Herbert N. Carlton, Nathan Altshiller, C. E. Horne, C. N. Schmall, Paul Capron, J. A. Caparo, Frank Irwin, Frank R. Morris, S. A. Joffe, V. M. Spunar, L. G. Weld, George W. Hartwell, Elijah Swift, A. W. Smith, Joseph B. Reynolds, Richard Morris, Walter C. Eells, Albert N. Nauer, Elmer Schuyler, and A. M. Harding.

428. Proposed by FRANK IRWIN, University of California.

If the roots of the equation

$$x^n - na_1x^{n-1} + \binom{n}{2}a_2x^{n-2} + \cdots = 0$$

are all real, the condition that they should all be equal is $a_1^2 = a_2$. A proof of the sufficiency of the condition is readily obtained from a consideration of derivatives. A proof is desired not based on such considerations.

Solution by Elijah Swift, University of Vermont.

We are to show that if the roots are all real, and if $a_1^2 = a_2$, the roots are all equal. For that purpose write the above equation in the form

$$(x - a_1)^n + \binom{n}{2}(a_2 - a_1^2)x^{n-2} + \varphi(x) = 0,$$

where $\varphi(x)$ is a polynomial of degree $n - 3$ at most. Suppose $a_2 = a_1^2$, and write $y = x - a_1$. The equation becomes

$$y^n + \varphi(y + a_1) \equiv y^n + c_3 y^{n-3} + \cdots + c_n = 0.$$

If all the c's are zero, the theorem is true. If not, let c_k be the first coefficient not zero, and c_e the last. The equation is then

$$y^n + c_k y^{n-k} + \cdots + c_e y^{n-l} = 0.$$

Apply Descartes's rule to this equation. If we call the left hand side $f(y)$, $f(y)$ and $f(-y)$ can have together at most $n - k + 1 - l$ variations of sign (k odd), or $n - k + 2 - l$ (k even), hence at most $n - 2 - l$. The equation has, therefore, at least $2 + l$ zero or complex roots. Exactly l roots are zero, however, hence it must have 2 complex. This contradicts our hypothesis that the roots were all real. Hence all the c's are zero and $f(y) \equiv y^n$, and the equation in x is $(x - a_1)^n = 0$.

Also solved by LAENAS G. WELD.

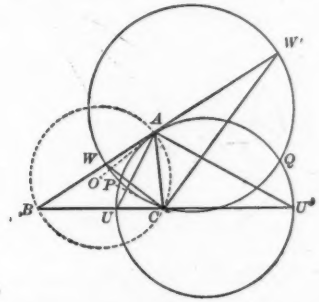
GEOMETRY.

456. Proposed by J. W. CLAWSON, Ursinus College.

The interior and exterior bisectors of the angles A, B, C of a triangle meet the opposite sides in U, U'; V, V'; W, W' respectively. Circles are drawn on UU', VV', WW' as diameters (Circles of Apollonius). Prove that (1) these three circles have a common chord. (2) The center of the circumcircle lies on this common chord.

SOLUTION BY PROPOSER.

1. $A(BC, UU')$ is a harmonic pencil; so (BC, UU') is a harmonic range. If T be any point on the circle having UU' as diameter, $T(BC, UU')$ is a harmonic



pencil. But UTU' is a right angle. Therefore TU bisects $\angle BTC$. Hence,

$$BT : TC = BU : UC = BA : AC.$$

In particular, if the circles having UU' and WW' as diameters intersect at the points P and Q,

$$BP : PC = BA : AC \text{ and } BQ : QC = BA : AC.$$

Similarly,

$$AP : PB = AC : CB \text{ and } AQ : QB = AC : CB.$$

Hence,

$$AP : PC = AB : BC \text{ and } AQ : QC = AB : BC.$$

It follows that P and Q are points on the circle having VV' as diameter. Hence the three circles UU', VV', WW' are coaxial.

2. Since (BC, UU') is a harmonic range, any circle passing through B and C is orthogonal to the circle having UU' as diameter. Hence, the circumcircle is orthogonal to that circle. Similarly the circumcircle is orthogonal to the circles VV' and WW'.

Hence, if O be the circumcenter, OA is a tangent to UU', OB to VV' and OC to WW'. But these tangents are equal in length.

Hence O is a point on the common chord of the circles UU', VV', WW'.

Also solved by C. E. Horne, David F. Kelley, Elijah Swift, Roger A. Johnson, and C. N. Schmall.

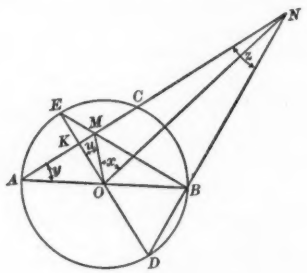
457. Proposed by NATHAN ALTSHILLER, University of Washington.

AB and *AC* are respectively a diameter and a chord of a circle whose center is *O*. The lines joining *B* to the extremities of the diameter perpendicular to *AC*, meet *AC* in the points *M*, *N*. Express the angle *MON* in terms of the angle *CAB*.

Solution by J. A. Caparo, University of Notre Dame.

Let E and D be the extremities of the diameter perpendicular to AC at the point K. With the notation of the figure we easily see that

$$\sin \angle AMB = \sin \angle ABN = \cos z.$$



Hence, by the sine law, letting $AB = 2R$, we have in $\triangle ANB$, $AN = 2R \cot z$, and in $\triangle AMB$, $AM = 2R \tan z$. Also from $\triangle AKO$, $AK = R \cos y$, $OK = R \sin y$. $KN = AN - AK = 2R \cot z - R \cos y$, $MK = AM - AK = 2R \tan z - R \cos y$.

$$\therefore \tan u = \frac{MK}{OK} \frac{2 \tan z - \cos y}{\sin y}. \quad \text{Also } \tan (x + u) = \frac{KN}{KO}.$$

Hence, substituting the values of KN and OK,

$$\frac{\tan x + \tan u}{1 - \tan x \tan u} = \frac{2 \cot z - \cos y}{\sin y}.$$

Substituting the value of $\tan u$,

$$2 \sin y(\tan z - \cot z) = 2 \tan x \cos y(\cot z + \tan z) - 5 \tan x.$$

But $\tan z - \cot z = -2 \tan y$ and $\cot z + \tan z = \frac{2}{\cos y}$.

Hence, substituting and reducing,

$$\tan x = 4 \tan y \sin y,$$

the required relation.

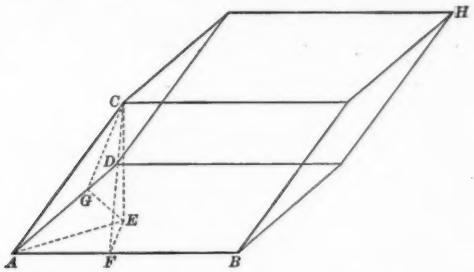
Also solved by PAUL CAPRON and J. W. CLAWSON.

458. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

Given edges l, m and n of a parallelopiped and angles a, b and c which the edges make with one another. Show that, if $s = \frac{a+b+c}{2}$, the volume equals

$$2lmn\sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}.$$

SOLUTION BY FRANK R. MORRIS, Glendale, Cal.



Given the parallelopiped AH with $\angle BAC = a$, $\angle DAC = b$, $\angle BAD = c$, and $AB = l$, $AC = m$, $AD = n$.

The area of the base BD is equal to $ln \sin C$.

From the vertex C drop a perpendicular to the base BD meeting it at E. From E draw perpendiculars to AB and AD meeting the lines at F and G, respectively. CE is the altitude of the parallelopiped and we know that the volume is $V = CE \cdot ln \sin c$. (1).

The triangles AEC, AFC, AGC, AFE and AGE are right triangles. Hence, we have

$$CE^2 = m^2 - AE^2, \quad (2) \qquad AF = m \cos a, \; AG = m \cos b, \quad (3)$$

$$\cos \angle EAF = \frac{AF}{AE}, \quad (4) \qquad \text{and} \qquad \cos (c - \angle EAF) = \frac{AG}{AE},$$

or

$$\cos c \cos \angle EAF + \sin c \sqrt{1 - \cos^2 \angle EAF} = \frac{AG}{AE}. \qquad (5)$$

Eliminating $\cos \angle EAF$ from (4) and (5)

$$\cos c \frac{AF}{AE} + \sin c \sqrt{1 - \frac{AF^2}{AE^2}} = \frac{AG}{AE}.$$

From this equation we get

$$AE^2 = \frac{AG^2 - 2AG \cdot AF \cos c + AF^2}{\sin^2 c}.$$

Substituting the values of AF and AG from (3),

$$AE^2 = \frac{m^2 \cos^2 b - 2m^2 \cos a \cos b \cos c + m^2 \cos^2 a}{\sin^2 c}.$$

Then from (2),

$$\begin{aligned}
CE^2 &= \frac{m^2}{\sin^2 c}(\sin^2 c - \cos^2 b + 2 \cos a \cos b \cos c - \cos^2 a) \\
&= \frac{m^2}{\sin^2 c}(\sin^2 c + \cos^2 c - \cos^2 b - \cos^2 c + \cos^2 b \cos^2 c - \cos^2 b \cos^2 c \\
&\qquad + 2 \cos a \cos b \cos c - \cos^2 a) \\
&= \frac{m^2}{\sin^2 c}\{(1 - \cos^2 b - \cos^2 c + \cos^2 b \cos^2 c) - \cos^2 a + 2 \cos a \cos b \cos c - \cos^2 b \cos^2 c\} \\
&= \frac{m^2}{\sin^2 c}\{(1 - \cos^2 b)(1 - \cos^2 c) - \cos^2 a + 2 \cos a \cos b \cos c - \cos^2 b \cos^2 c\} \\
&= \frac{m^2}{\sin^2 c}\{\sin^2 b \sin^2 c - \cos^2 a + 2 \cos a \cos b \cos c - \cos^2 b \cos^2 c\} \\
&= \frac{m^2}{\sin^2 c}\{\sin b \sin c + (\cos a - \cos b \cos c)\}\{\sin b \sin c - (\cos a - \cos b \cos c)\} \\
&= \frac{m^2}{\sin^2 c}\{\cos a - \cos (b + c)\}\{\cos (b - c) - \cos a\} \\
&= \frac{m^2}{\sin^2 c}\left\{2 \sin\left(\frac{a+b+c}{2}\right) \sin\left(\frac{b+c-a}{2}\right)\right\}\left\{\sin\left(\frac{a-b+c}{2}\right) \sin\left(\frac{a+b-c}{2}\right)\right\} \\
&= \frac{4m^2}{\sin^2 c} \sin s \cdot \sin (s - a) \sin (s - b) \sin (s - c),
\end{aligned}$$

where

$$s = \frac{a + b + c}{2}.$$

Hence, $CE = \frac{2m}{\sin c}\sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s - c)}$. Substituting this value of CE in (1), we have

$$v = 2lmn\sqrt{\sin s \sin (s - a) \sin (s - b) \sin (s - c)}.$$

Also solved by GEORGE W. HARTWELL, HORACE OLSON, J. A. CAPARO, and J. W. CLAWSON.

CALCULUS.

370. Proposed by PAUL CAPRON, Annapolis, Maryland.

The surface of a right circular cone having the semi-vertical angle α is cut by two planes, which intersect the axis at the same point, one at right angles to the axis, the other making the angle $(90° - \beta)$ with the axis. Show that if the lateral surface of the right cone is S_1 and that of the oblique cone S_2,

$$S_2 = \sum_1^\infty T_n, \qquad \text{where} \qquad T_1 = S_1,\ T_{n+1} = T_n \times \frac{2n+1}{2n}(\tan \alpha \tan \beta)^2.$$

SOLUTION BY THE PROPOSER.

Let the vertex of the cone be the origin, its axis the z-axis, and let the intersection of the oblique plane with the xy-plane be parallel to the y-axis. Let the two planes cut the z-axis at $(0, 0, h)$, and let the radius of the right section be a. Then,

$$\sqrt{x^2+y^2} = r, \text{ and } y/x = \tan\theta.$$

We have the following equations:

For the cone, $z = k \cot\alpha$, and for the oblique plane, $z = h - x\tan\beta x$. Eliminating z, we have for the projection on the xy-plane of the oblique section,

$$r = \frac{h\tan\alpha}{1+\tan\alpha\tan\beta\cos\theta}.$$

If dA is an element of area in the xy-plane, the element of the conical surface of which it is the projection is $\csc\alpha \cdot dA$. Hence $dS_2 = \csc\alpha\, rd\theta dr$, and

$$S_2 = \tfrac{1}{2}h^2\csc\alpha\tan^2\alpha\int_0^{2\pi}\frac{d\theta}{(1+\tan\alpha\tan\beta\cos\theta)^2}$$

$$= \frac{h^2}{2}\sec\alpha\tan\alpha\int_0^{2\pi}(1+k\cos\theta)^{-2}d\theta,$$

where

$$k = \tan\alpha\tan\beta.$$

$$\int_0^{2\pi}(1+k\cos\theta)^{-2}d\theta = \int_0^{2\pi}(1-2k\cos\theta+3k^2\cos^2\theta-\cdots)d\theta$$

$$= 2\pi\left[1+\frac{3}{2}k^2+\frac{15}{8}k^4+\frac{35}{16}k^6+\cdots\right],$$

where the general term is

$$T_{n+1} = (2n+1)k^{2n}\int_0^{2\pi}\cos^{2n}\theta d\theta = \frac{(2n+1)k^{2n}(2n-1)(2n-3)\cdots 1}{2n(2n-2)\cdots 2}\frac{\pi}{2}\times 4.$$

Hence,

$$T_{n+1} = T_n \times \frac{2n+1}{2n}k^2,$$

where

$$k = \tan\alpha\tan\beta.$$

$$T_1 = \pi k^2\tan\alpha\sec\alpha = \pi a^2\csc\alpha = S_1.$$

Also solved by ALBERT N. NAUER.

371. Proposed by B. F. FINKEL, Drury College.

Prove that the shortest distance between two curves or surfaces is normal to each.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

Let us prove this for two surfaces; the proof for two curves is similar. Suppose the equations of the surfaces are $f(x, y, z) = 0$, $F(x, y, z) = 0$, and that the shortest distance is between the points (x_1, y_1, z_1) on f and (x_2, y_2, z_2) on F. Then we wish to make $(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2$ a minimum with the auxiliary conditions $f(x_1, y_1, z_1) = 0$, $F(x_2, y_2, z_2) = 0$. Necessary conditions are that the partial derivatives of

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 + \lambda f(x_1, y_1, z_1) + \mu F(x_2, y_2, z_2),$$

where λ and μ are constants, shall all vanish. This gives us six equations, of which these are types:

$$2(x_1 - x_2) + \lambda \frac{\partial f}{\partial x_1} = 0,$$

$$-2(x_1 - x_2) + \mu \frac{\partial F}{\partial x_2} = 0.$$

From these, we have at once

$$\frac{\partial f}{\partial x_1} : \frac{\partial f}{\partial y_1} : \frac{\partial f}{\partial z_1} = x_1 - x_2 : y_1 - y_2 : z_1 - z_2.$$

But $\partial f/\partial x_1$, $\partial f/\partial y$, $\partial f/\partial z_1$, are proportional to the direction cosines of the normal to $f(x_1, y_1, z_1) = 0$ at (x_1, y_1, z_1). And $x_1 - x_2$, $y_1 - y_2$, $z_1 - z_2$, are proportional to the direction cosines of the line joining (x_1, y_1, z_1) and (x_2, y_2, z_2). Hence this line coincides with the normal. Similarly it is normal to $F(x_1, y_1, z_1) = 0$. Hence, the shortest distance must be perpendicular to both surfaces, *but not necessarily conversely.*

372. Proposed by V. M. SPUNAR, Chicago, Illinois.

Find the condition that the equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \left(1 + \frac{a^2}{x^2}\right)y = 0$$

should have one solution expressible in integral powers of x; and show that if this condition is satisfied, every other solution of the equation possesses a logarithmic infinity at the origin.

SOLUTION BY ALBERT N. NAUER, St. Louis, Mo.

Let $y = vx^{-\frac{1}{2}}$. Then finding the first and second derivatives of y with respect to x and substituting these values in the equation, we have

$$\frac{d^2v}{dx^2} - v = \left(\frac{a^2 - \frac{1}{4}}{x^2}\right)v.$$

This equation is of the form

$$\frac{d^2u}{dx^2} - \alpha^2 u = \frac{m(m+1)}{x^2}u,$$

where $m(m+1) = a^2 - \frac{1}{4}$, or $m = -\frac{1}{2} \pm a$ and $\alpha = 1$. This equation has the three following known solutions:

(1) $$u = x^{-m}\left(1 - \frac{1}{m - \frac{1}{2}}\frac{\alpha^2x^2}{2^2} + \frac{1}{(m-\frac{1}{2})(m-\frac{3}{2})}\frac{\alpha^4x^4}{2!\,2^4} - \frac{1}{(m-\frac{1}{2})(m-\frac{3}{2})(m-\frac{5}{2})}\frac{\alpha^6x^6}{3!\,2^6} + \cdots\right),$$

(2) $$u = e^{\alpha x}x^{-m}\left(1 - \frac{m}{m}\alpha x + \frac{m(m-1)}{m(m-\frac{1}{2})}\frac{\alpha^2x^2}{2!} + \frac{m(m-1)(m-2)}{m(m-\frac{1}{2})(m-1)}\frac{\alpha^3x^3}{3!} - \cdots\right),$$

(3) $$u = e^{-\alpha x}x^{-m}\left(1 + \frac{m}{m}\alpha x + \frac{m(m-1)}{m(m-\frac{1}{2})}\frac{\alpha^2x^2}{2!} - \frac{m(m-1)(m-2)}{m(m-\frac{1}{2})(m-1)}\frac{\alpha^3x^3}{3!} + \cdots\right).$$

Solution (2) is a finite primitive solution when m is an integer. When m is not an integer or zero, solution (1) gives an infinite series in powers of x. To have integral powers of x, m must equal $-(2n+1)/2$ when n is any integer or zero; also in the original equation $\pm a$ must equal $(2n+3)/2$.

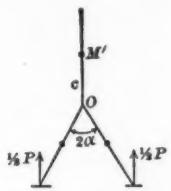
MECHANICS.

293. Proposed by B. F. FINKEL, Drury College.

A man of weight w stands on smooth ice; prove that if, when he gradually parts his legs, kept straight, with his feet in contact with the ice, the pressure of his feet on the ice be constant, his head will descend with uniform acceleration; and that, if f be the acceleration of his head, when his feet exert no pressure on the ice, their pressure on the ice, if f' were the acceleration of his head, would be equal to $\frac{f-f'}{f}w$. Walton's *Problems in Theoretical Mechanics*, p. 662.

SOLUTION BY E. B. WILSON, Massachusetts Institute of Technology.

We may analyze the man's total mass, M, into M'', the mass of the legs, and M', the remaining mass. The forces acting are W down and P up. Let 2α be the angle between the legs, l their length, $a\cos\alpha$ the distance of their center of gravity below O. Let c be the distance of the center of gravity of M' above O. The head falls through the distance $l(1-\cos\alpha)$. The center of gravity of the whole mass is at a height



$$h = \frac{M'(c + l\cos\alpha) + M''(l-a)\cos\alpha}{M = M' + M''}$$

above the ice. Hence the downward acceleration of the head is

$$f' = -l\frac{d^2\cos\alpha}{dt^2}.$$

The downward acceleration of his C.G. is

$$-\frac{M'l + M''(l-a)}{M}\frac{d^2\cos\alpha}{dt^2} = \frac{W-P}{M}.$$

Hence,

$$\frac{M'l + M''(l-a)}{M}\frac{f'}{l} = \frac{W-P}{M}.$$

If P is constant, then f', which is the only possible variable in this equation, must also be constant. This proves the first part.

If f be the value of f' when $P = 0$, we have

$$\frac{M'l + M''(l-a)}{M}\frac{f}{l} = \frac{W}{M}.$$

Hence,

$$\frac{P}{M} = \frac{M'l + M''(l-a)}{M}\frac{(f-f')}{l}$$

and

$$\frac{P}{W} = \frac{f-f'}{f}.$$

This proves the second part.

294. Proposed by EMMA GIBSON, Student at Drury College.

A sphere, revolving about a diameter and not acted on by any extraneous force, expands symmetrically; prove that its vis viva varies inversely as its moment of inertia about its diameter.

SOLUTION BY E. B. WILSON, Massachusetts Institute of Technology.

The moment of momentum of the sphere is $I\omega$, where I is the moment of inertia and ω the angular velocity about the axis. This is constant as no external forces are acting. The kinetic energy is $\frac{1}{2}I\omega^2$ or $I^2\omega^2/2I$, which proves the proposition.

MECHANICS.

295. Proposed by B. F. FINKEL, Drury College.

A homogeneous hollow cylinder, whose inner radius is half of its outer radius, rolls without slipping down a plane inclined at an angle α to the horizon. Find its acceleration.

I. SOLUTION BY A. M. HARDING, University of Arkansas.

The external forces acting are W pounds at the center vertically downward, the reaction normal to the plane, and the friction up the plane.

Let R denote the resultant of the last two and let β denote the angle that its direction makes with the normal. Then the equation of motion of the mass center is

$$W\frac{d^2s}{dt^2} = Wg\sin\alpha - R\sin\beta. \tag{1}$$

If the length of the outer radius of the cylinder is a, then

$$I\frac{d^2\theta}{dt^2} = R\,x\,a\sin\beta = Ra\sin\beta,$$

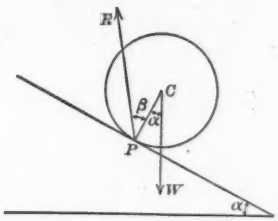
where the moment of inertia of the cylinder about its axis is $I = \frac{5}{8}Wa^2$.

But, since the cylinder does not slide,

$$s = a\theta. \quad \therefore \frac{d^2s}{dt^2} = a\frac{d^2\theta}{dt^2}.$$

Hence,

$$I\frac{d^2s}{dt^2} = Ra^2 \sin\beta, \qquad \text{or} \qquad R\sin\beta = \frac{I}{a^2}\cdot\frac{d^2s}{dt^2} = \frac{5W}{8}.$$



Substituting in equation (1), we obtain

$$\left(W + \frac{5}{8}W\right)\frac{d^2s}{dt^2} = Wg\sin\alpha. \quad \therefore \frac{d^2s}{dt^2} = \frac{8}{13}g\sin\alpha.$$

That is, 5/13 of the acceleration of gravity is used in turning the cylinder.

II. Solution by J. H. Kellogg, Oberlin College.

Let M, I, R and r be the mass, moment of inertia, and outer and inner radii of the cylinder respectively; $P.E.$ and $K.E.$ the changes in potential and kinetic energy, due to the descent; v the linear velocity, ω the angular velocity, and a the acceleration, of a point on the outer circumference; d the length of the plane; and g the acceleration due to gravity.

Remembering that the $K.E.$ has two parts, one translational and the other rotational; and that

$$R = 2r, \qquad v = r\omega, \qquad v^2 = 2ad, \qquad \text{and} \qquad I = M\frac{R^2 + r^2}{2};$$

then

$$K.E. = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}Mv^2\left[1 + \frac{R^2 + \left(\frac{R}{2}\right)^2}{2R^2}\right] = \frac{13}{16}Mv^2 = \frac{13}{8}Mda,$$

and

$$P.E. = Mg\,(d\sin\alpha).$$

But the energy of the system must remain the same; hence

$$P.E. = K.E.$$

and

$$a = \frac{8}{13}g\sin\alpha.$$

NUMBER THEORY.

224. Proposed by PATRICK WALSH, New Orleans, Louisiana.

Find the sides, in rational numbers, of a right angled triangle whose area is $5\frac{1}{2}$.

SOLUTION BY C. E. HORNE, Westminster, Colorado.

Let $11x/y$ and y/x be the legs of the triangle, thus fulfilling one condition of the problem. Then

$$\frac{121x^2}{y^2}+\frac{y^2}{x^2}=\frac{121x^4+y^4}{x^2y^2}=\square.$$

This will be so when $121x^4+y^4=\square$. Let $\sqrt{121x^4+y^4}=11x^2+zy^2$. Then

$$x=y\sqrt{\frac{1-z^2}{22z}}$$

and

$$\frac{x}{y}=\sqrt{\frac{1-z^2}{22z}},$$

will be rational when

$$\sqrt{\frac{1-z^2}{22z}}=\square=\text{say } t^2.$$

Solving for z, we have $z=-11t^2\pm\sqrt{121t^4+1}$. When $t=3/70$ $z=-50/49$ or $49/50$. Hence, $x/y=3/70$; $11x/y=33/70$ and $y/x=70/3$. The hypotenuse $=4901/210$.

NOTE.—Having found one value of t which makes $\sqrt{121t^4+1}$, rational, we may find other values by making use of Euler's method, in his *Elements of Algebra*, third edition, revised by Hewlett, Chapter IX, p. 374.—EDITOR.

225. Proposed by W. DE W. CAIRNS, Oberlin College.

L'Intermédiaire for June, 1914, contains the following problem: "If we write the terms of the arithmetic series 1, 5, 9, 13, 17, 21, 25, 29, 33, ··· as follows:

$$\begin{array}{ccccccccc}
1 & & & & & & & & \\
5 & & 9 & & 13 & & & & \\
17 & & 21 & & 25 & & 29 & & 33 \\
37 & & 41 & & 45 & & 49 & & 53 \quad 57 \quad 61 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot,
\end{array}$$

it is seen that the sum of the terms of each line is a cube, and that these are the cubes of the successive odd integers. How is this shown?"

It is here proposed not only to prove this, but to generalize the theorem as suggested, using, however, the simpler (and better known) case which includes all of the successive integers:

$$\begin{array}{ccccccc}
1 & & & & & & \\
3 & & 5 & & & & \\
7 & & 9 & & 11 & & \\
13 & & 15 & & 17 & & 19 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \quad \cdot
\end{array}$$

SOLUTION BY THOS. E. MASON, Purdue University.

Theorem. Form the arithmetic series $1+2kn$, $n=0, 1, 2, 3, \cdots$, where k is any positive integer which remains constant for a given series. Arrange this

series in rows, 1 in the first row, and each row after the first containing k more terms than the preceding. Then the sum of the terms in any row is the cube of the number of terms in that row.

From the way in which the series is formed, the value of n for any term will be the same as the number of terms preceding it in the series. The number of terms in the rows will form an arithmetic series with first term 1 and difference k. Making use of these facts, we obtain the following:

The number of terms in the rth row is $1 + (r - 1)k$.

The value of n for the first term in the rth row is the sum of $r - 1$ terms of the arithmetic series with first term 1 and difference k, that is,

$$\frac{r-1}{2}[2 + (r - 2)k].$$

Making use of this value of n, we have for the first term in the rth row the value $1 + k(r - 1)[2 + (r - 2)k]$. The sum of the terms in the rth row will be the sum of $1 + (r - 1)k$ terms of the arithmetic series with first term $1 + k(r - 1)[2 + (r - 2)k]$ and difference $2k$, that is,

$$\frac{1 + (r-1)k}{2}\{2[1 + k(r - 1)\{2 + (r - 2)k\}] + k(r - 1)2k\} = \{1 + (r - 1)k\}^3.$$

This proves the theorem.

The two arrangements of the problem can be obtained by making $k = 2$ and $k = 1$, respectively.

Also solved by R. M. MATHEWS, ELIJAH SWIFT, HARMON L. SLOBIN, and S. A. JOFFE.

QUESTIONS AND DISCUSSIONS.

EDITED BY U. G. MITCHELL, University of Kansas.

At the time of making up copy for this issue further replies are desired to questions numbered 4, 8, 12, 13, 16, 20, 23, 24, 25 and 26.

NEW QUESTIONS.

27. A certain college wishes to offer twelve hours of mathematics beyond the usual courses in analytical geometry and differential and integral calculus. Considering only the needs of students intending to specialize in pure mathematics, what courses should make up the twelve hours offered?

28. Is it possible to obtain $\int \cos \theta^2 d\theta$ without expanding $\int \cos \theta^2$? If it is not, can some interesting properties of this integral be determined by treating it as a special function?

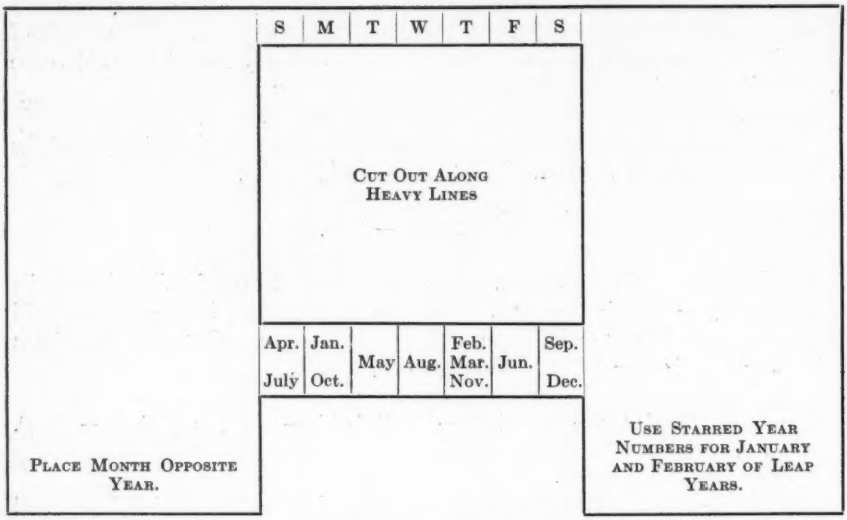
DISCUSSIONS.

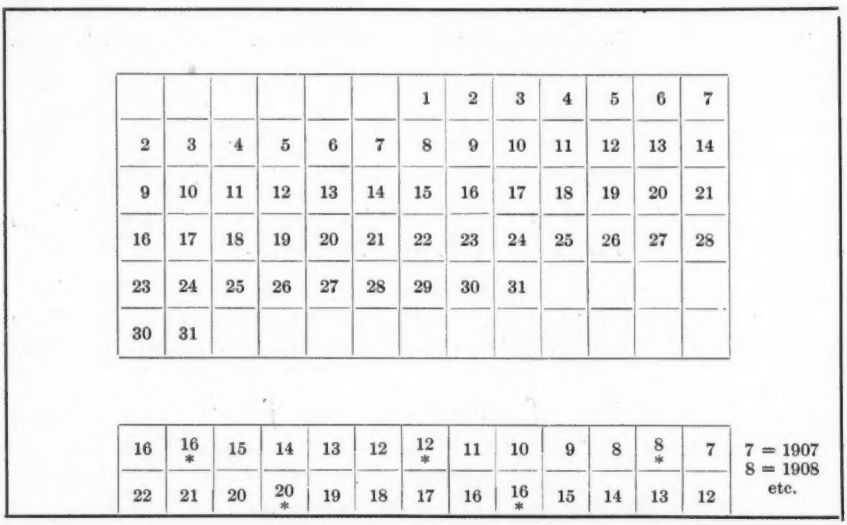
RELATING TO ADJUSTABLE CALENDARS.

BY IRWIN ROMAN, Chicago, Ill.

So far as the writer has been able to learn, all perpetual or adjustable calendars are arranged so as to present the first day of the month as the first day of the week.

This arrangement makes it necessary to look at the top of the calendar to see what day of the week a certain date is. The accompanying drawings illustrate a calendar which presents Sunday first as in the ordinary printed sheet calendar.





The figures are self-explanatory. To use the calendar, slide the top card over the bottom card until the name of the month and the number of the year desired are opposite each other. The calendar for that month will then appear

through the section cut from the top card. The details of construction and the reasons for the arrangements need not be given here, but will be furnished if requested. The period of years may be extended at will. The size and shape may also be varied to suit individual tastes.

RELATING TO SOLUTIONS OF QUADRATIC EQUATIONS.

By Geo. R. Dean, Missouri School of Mines.

I. *Solution of the Quadratic without factoring or completing the square.*

Let the equation be

$$ax^2 + bx + c = 0.$$

Put

$$x = u + iv, \qquad \text{where} \qquad i = \sqrt{-1}.$$

Then

$$a(u^2 - v^2 + 2uvi) + b(u + iv) + c = 0,$$

$$a(u^2 - v^2) + bu + c + i(2auv + bv) = 0,$$

Since the real and imaginary parts vanish separately,

$$a(u^2 - v^2) + bu + c = 0, \qquad \text{and} \qquad (2au + b)v = 0.$$

And since v is not, in general, equal to zero, we get

$$u = -\frac{b}{2a},$$

from which

$$au^2 + bu + c = c - \frac{b^2}{4a}.$$

Hence,

$$av^2 = c - \frac{b^2}{4a}, \qquad v = \frac{\pm\sqrt{4ac - b^2}}{2a};$$

$$u + iv = \frac{-b \pm i\sqrt{4ac - b^2}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

When v, that is, $\frac{\sqrt{4ac - b^2}}{2a}$ is imaginary the equation has real roots; and when $v = 0$, equal roots.

There is probably nothing new about this solution, but it affords a good example of the part played by the imaginary unit in higher mathematics, and would not be out of place in our elementary text-books on algebra.

II. *Solution of a Pair of Simultaneous Equations which occurs in the Theory of Cables and Transmission Lines.*

In the following equations the unknown quantities are α and β:

$$\alpha^2 - \beta^2 = RS - LCp^2 \quad (1); \qquad 2\alpha\beta = (RC + LS)p. \quad (2)$$

The solution by the regular algebraic process gives

$$\alpha = \sqrt{\tfrac{1}{2}\{\sqrt{(R^2 + p^2L^2)(S^2 + p^2C^2)} + (RS - LCp^2)\}}$$

$$\beta = \sqrt{\tfrac{1}{2}\{\sqrt{(R^2 + p^2L^2)(S^2 + p^2C^2)} - (RS - LCp^2)\}}.$$

A more elegant solution, from the mathematician's point of view, more convenient for the computer, and furnishing at the same time the value of three other quantities that are needed in other computations, is obtained by using trigonometric functions.

Let

$$\alpha = N\cos\xi, \beta = N\sin\xi, \text{ then } \alpha^2 - \beta^2 = N^2\cos 2\xi;$$

$$R = Z\cos\delta,\ pL = Z\sin\delta, \text{ then } \tan\delta = \frac{pL}{R},\ Z = \frac{R}{\cos\delta} = \frac{pL}{\sin\delta};$$

$$S = Y\cos\gamma,\ pC = Y\sin\gamma, \text{ then } \tan\gamma = \frac{pC}{S},\ Y = \frac{S}{\cos\gamma} = \frac{pC}{\sin\gamma};$$

$$RS - LCp^2 = ZY\cos(\gamma + \delta),\ (RC + LS)p = ZY\sin(\gamma + \delta).$$

Therefore

$$N^2\cos 2\xi = ZY\cos(\gamma + \delta),\ N^2\sin 2\xi = ZY\sin(\gamma + \delta),$$

and it is easy to see that

$$N = \sqrt{ZY}, \text{ and } \xi = \tfrac{1}{2}(\gamma + \delta).$$

As a numerical illustration, take $R = 0.30$, $L = 0.00196$, $C = 0.0153 \times 10^{-6}$, $S = 0$, $p = 377$. Then, using a slide rule,

$$\tan\delta = \frac{pL}{R} = 2.4600,\ \delta = 67^\circ\ 53',\ \cos\delta = 0.3765,\ Z = 0.795;$$

$$\tan\gamma = \frac{pC}{S} = \infty,\ \gamma = 90^\circ,\ \xi = \tfrac{1}{2}(\gamma + \delta) = 78^\circ\ 56'\ 30'',\ Y = 5.77 \times 10^{-6};$$

$$\alpha = \sqrt{ZY}\cos\xi = 0.000412,\ \beta = \sqrt{ZY}\sin\xi = 0.002100.$$

The quantities γ, δ, ξ are useful in other computations.

NOTES AND NEWS.

EDITED BY W. D. CAIRNS, Oberlin, Ohio.

Professor R. D. Carmichael has accepted a position in the University of Illinois.

Professor B. F. FINKEL was in charge of the mathematics at the summer school of the University of Colorado.

Mr. C. R. Dines, of Northwestern University, has been appointed to a position as instructor in mathematics at Dartmouth College.

Dr. H. B. PHILLIPS has been promoted to an assistant professorship of mathematics in the Massachusetts Institute of Technology.

Professor L. C. KARPINSKI, of the University of Michigan, and Mr. E. F. Gee- of Detroit Central High School, were re-elected chairman and secretary, respec, tively, of the mathematics section of the Michigan Schoolmasters' Club at its April, 1915, meeting.

Mr. A. J. MILLER, instructor in mathematics at the University of Michigan, has been appointed by Harvard University to a travelling fellowship for the year 1915–1916. Mr. Miller expects to spend the year at Turin, Italy, studying with Professor Segre.

An article by Professor W. H. ROEVER of Washington University, on "The design and theory of a mechanism for illustrating certain systems of lines of force and stream lines," appeared in a recent number of *Zeitschrift für Mathematik und Physik*.

The death is reported of William B. Graves, professor emeritus of natural science at Phillips Academy, Andover, Mass., and for seven years professor of mathematics and civil engineering at the Massachusetts Agricultural College.

A work by G. F. HILL, of the British Museum, on the spread of the Hindu numerals in Europe has recently been published by the Clarendon Press, Oxford. A review will appear in a later issue.

Mr. J. C. NICHOLS, assistant professor of mathematics at the Texas Agricultural College, has been appointed to a fellowship in mathematics at the University of Michigan for the year 1915–1916. Mr. Nichols will devote himself to work in the history of mathematics.

Mr. CARL COE, who has been studying in Harvard University for the past two years, has resumed his position as instructor of mathematics in the University of Michigan.

An Italian edition of the work on "Mathematical Recreations and Problems," by W. W. ROUSE BALL, was published in 1911 (Bologna, N. Zanichelli), translated by Professor D. Gambioli.

The publishing house of A. Formiggini in Genoa has in progress a series of popular biographies (*Profili*), published at one lira, including in the numbers already issued *Archimedes*, by Favaro, and *Galileo Galilei*, by the same author.

Dr. Roger A. Johnson, of Western Reserve University, gave a course on "Geometry for Teachers" at the Harvard University summer school this summer.

On the list of those elected to membership in the American Philosophical Society occur the names of Professor W. F. OSGOOD of Harvard University, and Professor J. A. MILLER of Swarthmore College.

At Harvard University Dr. E. V. HUNTINGTON has been promoted to an associate professorship in mathematics, and at the University of Minnesota Dr. W. H. BUSSEY has been likewise honored.

Mr. C. H. YEATTON has been appointed to an instructorship in mathematics at Northwestern University. He has just completed his work for the doctorate at the University of Chicago.

Dr. H. S. WHITE, professor of mathematics in Vassar College, and Dr. R. A. MILLIKAN, professor of physics in the University of Chicago, were elected to membership in the National Academy of Sciences on April 21, 1915.

Professor E. F. CODDINGTON, of the department of mechanics in Ohio State University, will be acting dean of the college of engineering this year during the absence of Dean Orton; Professor C. C. MORRIS, of the department of mathematics, will act as assistant to the dean.

Professor A. B. COBLE of Johns Hopkins University, and Professor W. A. HURWITZ of Cornell University, have been chosen associate editors of the *Transactions of the American Mathematical Society*.

JOSEPH J. HARDY, professor of mathematics and astronomy, and for forty-five years a member of the faculty of Lafayette College, died on May 2, 1915.

Professor HENRY SUZALLO, of the department of education in Teachers College, Columbia University, and author of "The Teaching of Primary Arithmetic" and other educational texts, has been elected to the presidency of the University of Washington.

In *School Science and Mathematics* for June Professor M. O. TRIPP, of Olivet College, writes on "Some simple applications of elementary algebra to arithmetic," and Mr. H. C. WRIGHT, in a paper on "Mathematical equipment and its uses" reports some of the methods employed in the University of Chicago high school.

The January–February number of *Rendiconti del Circolo Matematico di Palermo* contains two articles by Americans: "Continuity of functions of infinitely many variables," by Professor W. D. A. WESTFALL, of the University of Missouri, and "Infinite developments and the composition property $(K_{12}B_1)$ in general analysis," by Dr. E. W. CHITTENDEN, of the University of Illinois.

Mr. A. L. NELSON has been appointed to an instructorship in mathematics at the University of Michigan. Mr. Nelson is a graduate of the University of

Kansas and has just completed his work for the Doctorate at the University of Chicago.

Miss MARY E. WELLS has accepted a position as instructor in mathematics at Vassar College. During the past year she acted as supply at Oberlin College during the leave of absence of Dr. Mary E. Sinclair. Miss Wells has just taken the doctorate at the University of Chicago.

Dr. P. ZEEMAN, since 1902 professor of geometry and theoretical mechanics in the University of Leyden, died on May 8, 1915.

The *compte rendu* of the international conference on the teaching of mathematics held last April in Paris has now been published by the Georg press in Geneva, Switzerland, under the editorship of the secretary, Professor H. Fehr.

Mr. H. E. Webb, head of the department of mathematics in the Commercial and Manual Training High School in Newark, N. J., is chairman of a committee of the Society for the Promotion of Engineering Education, which has recently made a preliminary report on "Cooperation with secondary schools," which should be of interest to readers of the MONTHLY. It is printed in Volume XXII, 1915.

Dr. WILLIAM M. SMITH, of the University of Oregon, has been elected associate professor of mathematics at Lafayette College in place of the late Professor Hardy. Dr. Smith will also be registrar of the College.

Dr. MORGAN W. CROFTON, F.R.S., formerly professor of mathematics at Queen's College, Galway, and later professor of mathematics and mechanics at the Royal Military Academy, Woolwich, died on May 13, 1915, in his eighty-ninth year. He was the author of text-books and tracts on mechanics and also contributed papers to the leading mathematical journals. He wrote the chapter on mean values and probability in Williamson's "Integral Calculus," a topic in which he was a pioneer investigator.

The International Commission on the Teaching of Mathematics has recently issued, through the Bureau of Education at Washington, an important bulletin on the teaching of elementary and secondary mathematics in all the leading countries of the world. This bulletin, No. 45, 1914, was prepared by J. C. BROWN, and sets forth the nature of the mathematics taught in every school year, from the first through the twelfth, in the standard type of school. It should be in the hands of every teacher of mathematics, and may be secured by addressing the United States Commissioner of Education, Washington, D. C.

About eighty-five teachers of mathematics and allied sciences were present at the last meeting of the Michigan Schoolmasters' Club. The papers at this session were devoted entirely to practical phases of the teaching of high school mathematics. Correlation and real problems occupied a prominent place in the discussion. At the meeting for April 1916, it is proposed to continue the discussion of real problems in high school work, and also to discuss the relation of the various phases of higher mathematics to the elementary mathematics.

A new number of the *Encyclopédie des Sciences Mathématiques* was published recently. It is dated March 15, 1915, and is the first number of the French edition of this extensive work, which has appeared since the beginning of the present European War. It is an unusually large number, containing 292 pages, and is devoted to the principles of rational mechanics and to statistical mechanics. The article on the former of these two subjects is more than fifty per cent. larger than the corresponding article in the German edition, but the article on the latter subject is only slightly changed, except that a supplement is added bringing the work up to date.

In spite of war conditions, recent numbers of *L'Intermédiaire* have been sent to its subscribers. Although the character of problems proposed in this journal does not perhaps permit of their being offered in our department of Problems, it will be interesting to quote a few which are typical.

(1) Integrate the partial differential equation

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} + \frac{\partial^2 z}{\partial x \partial y} = 0.$$

(2) Form a table of the possible and impossible values for h, less than an assigned number, given that

$$x^3 - hy^3 = z^2, \quad x^3 + hy^3 = t^2,$$

x, y, z, t and h being positive integers. For example, $x = 5$, $y = 1$, $z = 9$, $t = 13$, $h = 44$.

(3) Show that

$$a^2 \int_0^{\pi/2} \frac{\sin^4\varphi \cos^2\varphi \, d\varphi}{(a^2 \sin^2\varphi + b^2 \cos^2\varphi)^4} - b^2 \int_0^{\pi/2} \frac{\sin^2\varphi \cos^4\varphi \, d\varphi}{(a^2 \sin^2\varphi + b^2 \cos^2\varphi)^4} = 0$$

without calculating the definite integrals.

An interesting conference was held at the University of Indiana in May, 1915. It was the second annual conference on "Educational Measurements." There were fifteen or more speakers, and the program shows that a serious effort was made to discover some standards by which to judge educational work. The proceedings have been published in pamphlet form. The first conference, in 1914, resulted in trying out the Courtis Arithmetic Tests in grades V to VIII in twenty Indiana City school systems. The results were sent to the Psychological Laboratory of Indiana University, where they were studied and arranged in comparative tables and charts. The University has issued a pamphlet covering the results of this study under the title "Arithmetic: A Cooperative Study in Educational Measurements."

The eleventh annual session of the Association of Ohio Teachers of Mathematics and Science was held at the Ohio State University, April 2 and 3, under the presidency of Professor C. C. Morris of Ohio State University. Professor E. H. Taylor of the Eastern Illinois State Normal gave two lectures on "Recent tendencies in the teaching of secondary mathematics," in which among other things he recommended the establishment in this country of the "continuation school" now being conducted in Germany as a school intermediate in character between

the American trade schools and colleges of engineering. Mr. C. F. Geeting of the Canton high school read a paper on "The first two years in high school mathematics." The association directed the appointment of a large committee, representing all sections of the state, which is to draft a bill for introduction in the state legislature substituting the metric for the British system; this committee is also to conduct an educational campaign for its popular adoption and to urge the teaching of the metric system as early as the second and third school years.

Some interesting developments in connection with mathematical clubs among high school students have come to the attention of the MONTHLY. A club at Hutchinson, Kan., has discussed such topics as Euclid and his work, The trisection of an angle, The squaring of the circle, The use of graphs, The slide rule, History of our numerals, History of logarithms, Life of Pythagoras, with different proofs of his theorems, and a debate on the question whether one year of algebra and one year of geometry should be required for graduation from the high school, the decision resulting in the affirmative.

A club at the Wendell Phillips high school in Chicago has discussed the following topics: Pascal's theorem on a set of secant lines, The relation between the circumcenter of a triangle, the inradius, and the distance between these centers, The collinearity of the center of the nine point circle, the centroid, the orthocenter, and the circumcenter of a triangle.

A club at the Hyde Park high school, Chicago, has also accomplished some remarkable results, an account of which will soon be published in *School Science and Mathematics*, where also was published last year an extended report of a club at the Shattuck School, Faribault, Minn.

In all these clubs the membership and attendance are purely voluntary and the enthusiasm runs high.

L'Enseignement Mathématique, Vol. XVII, Nos. 1 and 2, Jan. 15 and Mar. 15, 1915, contain the following articles which are of particular interest to teachers of mathematics: (1) *The problem of interpolation and Taylor's formula*, by R. Suppantschitsch of Vienna, (2) *On the trinomial of the second degree*, $ax^2 + bx + c$, by P. Suchar, of Pau; (3) *On the teaching of mathematics*, by G. Fontené, of Paris; and (4) *Trigonometry and its relations to geometry*, by A. Streit in Bern. In the article by Streit the theorems of Ceva, Menelaus, and Ptolemy are derived from simple trigonometrical and geometrical considerations.

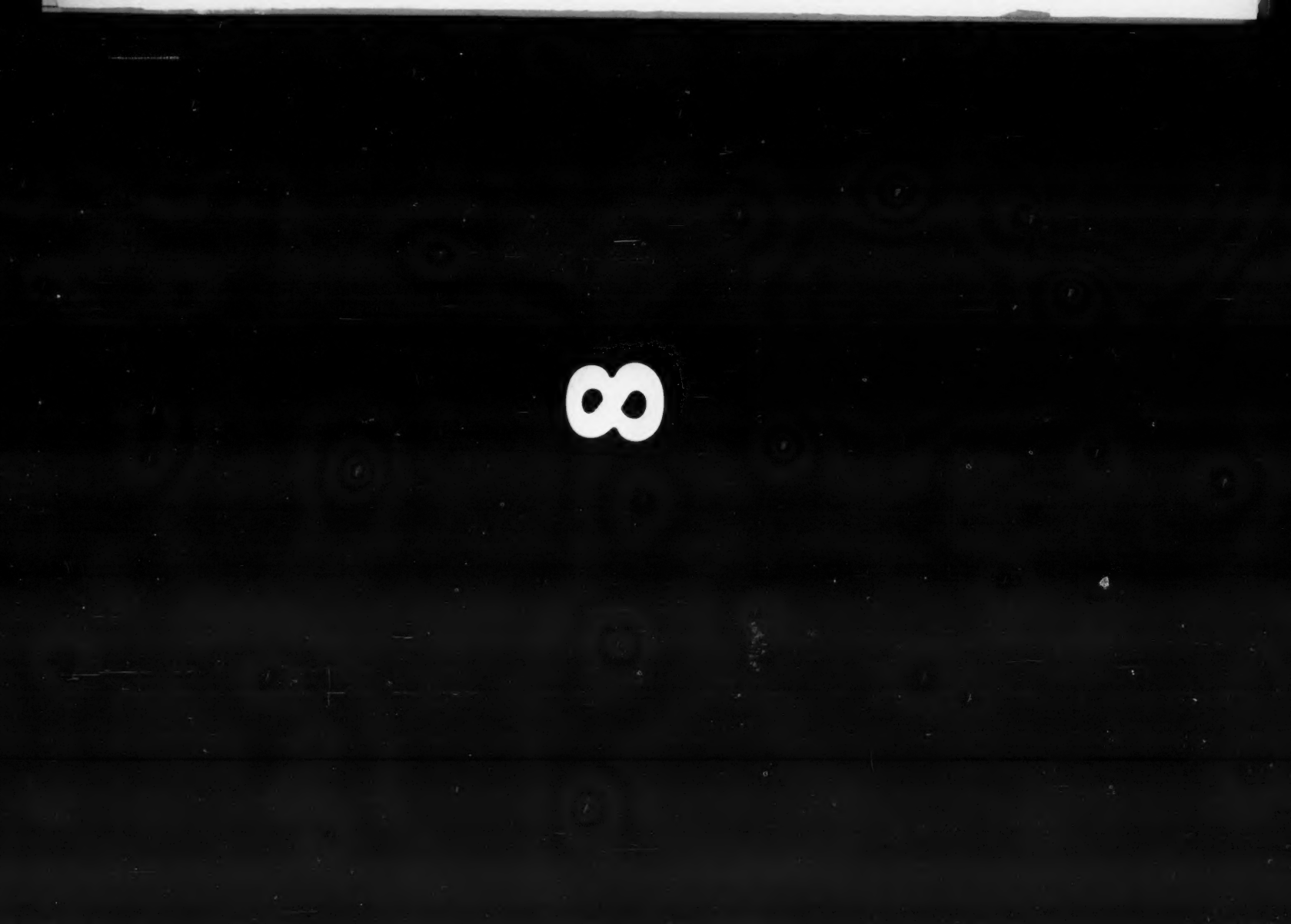
These numbers also give the questionnaire on an "Inquiry into the training of teachers of mathematics in secondary schools in different countries," as proposed by the Central Committee of the International Commission on the Teaching of Mathematics. "The Central Committee desires to continue its work, though renouncing the hope of summoning a conference. If the national subcommissions furnish the necessary documents, the work projected for 1915 will be collected in a pamphlet similar to that which was devoted to the conference at Paris." The parts of the questionnaire are devoted to (1) general preparation of candidates, (2) theoretical scientific teaching, (3) professional training, (4) subsequent improvement, (5) legal provisions as to teachers, (6) bibliography.

The *Bollettino di Bibliografia e Storia delle Scienze Matematiche*, a journal published under the editorship of Professor GINO LORIA, at Genoa, Italy, and devoted to the historical and bibliographical side of mathematics is now in its seventeenth year. The first three issues of 1914, the latest at hand, contain the following articles in Italian: (1) *Unpublished Contributions to the Correspondence of Evangelista Torricelli*, by Profesor Antonio Favaro; (2) *Concerning the Name, "Algoritmo,"* a discussion of the use of the word *Algorism* as a title and its recognition as part of the name of Mohammed ibn Musa al-Khowarizmi; and (3) *The Theory of Proportion in an Unpublished Manuscript of Evangelista Torricelli's*, by F. Podetti. This latter work connects directly with the attempts in the period of Galilei to explain in an easy way the substance of Euclid's definition of equal ratios and the rest of Book V, by means of the principles which Galilei wished to substitute for Euclid's definitions.

Each number contains, besides a leading article of the kind indicated bibliographical material topically distributed, as well as notes on current events of interest to mathematicians. Typographical errors in English, French and German quotations are fairly frequent. However, the journal certainly serves a useful purpose in stimulating an interest in the history of science, to which field Italians like COSSALI, LIBRI, LORIA, FAVARO and SCHIAPPARELLI have made most noteworthy contributions.

The following is from a letter of Professor Cajori written to the MONTHLY while he was traveling in Europe:

"I met in Zurich Professor F. Rudio, the editor-in-chief of the collected works of Leonhard Euler, which are now in course of publication. The war is causing some delay and other perplexing complications; nevertheless, the printing is progressing. I found Professor Rudio reading proof. He expressed appreciation of the contribution toward this enterprise of $1,000, made by the American Mathematical Society. I was disappointed when I learned that only about twenty subscriptions for the collected works came from the United States. Owing to lack of foresight in the management of our libraries, American mathematicians will not enjoy the convenience of easy access to Euler's works to a degree at all commensurate with the fine gift made by the American Mathematical Society. In view of the tremendous importance of this most prolific and genial eighteenth century mathematician, who figures in the early development of nearly every part of modern mathematics, the number of subscriptions from the United States should not be 20, but 220. The rapidly growing universities of the South, West and Middle West will find it later increasingly difficult and perhaps eventually impossible to secure the complete works of Euler. The edition will consist of 40 volumes. Subscriptions can still be secured through Professor F. Rudio (111 Dolderstrasse, Zurich) at the reduced rate of $5 per volume. As not more than three volumes are published per year, the yearly outlay is but slight."



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THE PROMOTION OF COLLEGIATE MATHEMATICS.

By H. E. SLAUGHT, University of Chicago.

At the meeting of the Chicago Section of the American Mathematical Society in April, 1914, there arose an informal discussion with respect to the field of collegiate mathematics. It was pointed out, on the one hand, that the interests of secondary mathematics have been well cared for in recent years both through the organization of effective secondary associations in all parts of the country, and through two representative secondary journals; and, on the other hand, that the interests of research in pure mathematics have been strongly fortified through the activities of the American Mathematical Society and its various publications, through the two research journals published under private university auspices, and through the multiplication of research fellowships at all the great universities.

In contrast to these two important phases of mathematical interests, it was pointed out that between them there is the great intermediate field of collegiate mathematics which so far has had no organized attention. No society is concerned particularly with this field and no journal represents its interests, except in so far as a few individuals have endeavored to do so in connection with the AMERICAN MATHEMATICAL MONTHLY. It was recognized that a large majority of the men and women in the mathematical faculties of the six hundred institutions of college grade in the United States are devoting themselves exclusively to teaching, and that very many even of those whose tastes and desires lie in the lines of research devote a good part of their time to teaching collegiate mathematics.

Furthermore, it appeared that mathematical students in our graduate schools come in large measure from smaller institutions in which the teachers are not members of the American Mathematical Society and have no professional contact with their colleagues in other institutions either by way of scientific investigation of through interchanging ideas on the many important questions still outstanding with respect to the collegiate curriculum in mathematics.

And, finally, it appeared that very many even within the present Society, and